G lcscu (unimodular) $\quad G \curvearrowright (X,\mu)$ pmp

Fix a Haar measure λ.

G nondiscrete

<u>Fact</u> Every free pmp action of such a G is isomorphic to a <u>point process action</u>.

<u>Setup</u> (Z,d) complete separable metric space

Typically, $Z = G$, $G/\text{compact}$, G/H $\quad$ $H \leq_c G$ $\quad$ $\Gamma \curvearrowright \{0,1\}^\Gamma$, $\Gamma \curvearrowright [0,1]^\Gamma$

<u>proper</u> left invariant metrics

closed balls are compact

Defn The *configuration space* of Z is $\mathbb{M} = \mathbb{M}(Z)$

$$\mathbb{M} = \{\omega \subset Z \mid \omega \text{ is locally finite}\}$$

w/ the σ-algebra generated by the functions

$$N_A : \mathbb{M} \to \mathbb{N}_0 \cup \{\infty\} \quad \text{for } A \subseteq Z \text{ Borel},$$

$$N_A(\omega) = |\omega \cap A|$$

A *point process* is a random element Π of $\mathbb{M}$, i.e. there is some probability space $(\Omega, \mathbb{P})$

and $\Pi : (\Omega, \mathbb{P}) \longrightarrow \mathbb{M}$ is measurable

Its _law_ or _distribution_ is $\Pi_*(\mathbb{P}) \in \mathrm{Prob}(\mathbb{M})$

Example Take $A \subseteq G$ measurable s.t. $0 < \lambda(A) < \infty$

$$\Pi_n : A^n \longrightarrow \mathbb{M}(G), \quad (a_1, \dots, a_n) \mapsto \{a_1, \dots, a_n\}$$

Equip A with $\dfrac{\lambda(\cdot \cap A)}{\lambda(A)} \in \mathrm{Prob}(A)$

Take N to be a random natural. Now take

$$\Pi : \mathbb{N} \times A^{\mathbb{N}} \longrightarrow \mathbb{M}$$

$$(n, (a_1, a_2, \dots)) \mapsto \{a_1, a_2, \dots, a_n\}$$

For Ω on G or G/H, we say Ω is invariant if $g\Omega \overset{(d)}{=} \Omega$, i.e. $\underline{G \curvearrowright (\mathbb{M},\mu)}$ is pmp.

"equal in dist."

Its <u>intensity</u> is
$$\mathrm{int}(\Omega) = \frac{\mathbb{E}[N_A(\Omega)]}{\lambda(A)}$$
where $0<\lambda(A)<\infty$.

gf_1

gf_2

$\overset{\times}{g}$ $\vdots$

gf_n

<u>Note</u> $A \mapsto \mathbb{E}[N_A(\Omega)]$ is a Haar measure

$$\mathbb{E}[N_A(\Omega)] = \mathbb{E}[N_A(g\Omega)]$$

take $F=\{e,\gamma\}$ with $\gamma\in\Gamma$ $\Omega F = \Omega$

Fix

ΩU

G

<u>Exa</u>

To

Un

int

Fix $F \subseteq G$ finite, ~~$e \in F$~~. Given invariant Ω, take $\Omega \cup \Omega F$. Its intensity is $\mathrm{int}(\Omega) \cdot |F|$ if G is unimodular

<u>Example</u> $\Gamma < G$ a lattice. Note $G/\Gamma \subseteq IM(G)$

To sample, fix $\mathcal{F}$ fund domain. Choose $a \in \mathcal{F}$ uniformly and take $a\Gamma$.

$$\mathrm{int}(G/\Gamma) = \frac{1}{\mathrm{covol}(\Gamma)} = \frac{1}{\lambda(\mathcal{F})}$$

$$\boxed{\mathrm{cost}(G/\Gamma) - 1 = \frac{d(\Gamma) - 1}{\mathrm{covol}(\Gamma)}}$$

We say Ω is <u>Poisson</u> with intensity $t > 0$ if

(I) $N_A(\Omega) \sim \mathrm{Pois}(t \cdot \lambda(A))$, concretely

$$\mathbb{P}[N_A(\Omega) = k] = e^{-t\lambda(A)} \cdot \frac{(t\lambda(A))^k}{k!}$$

$\mathrm{Prob}(\mathbb{N})$

(II) If $A \cap B = \emptyset$, then $N_A(\Omega)$ is independent of $N_B(\Omega)$

<u>Fact</u> [Rényi] (I) and (II) are equivalent

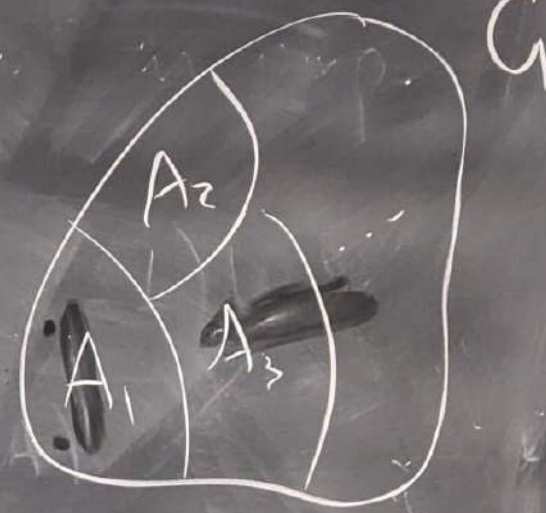
<u>Thm</u> [Wilkens] The <u>Poisson actions</u> are isomorphic

$$G \curvearrowright (M, \mathbb{P}_t)$$

__To construct__ Fix a partition $\mathcal{P}$ of G

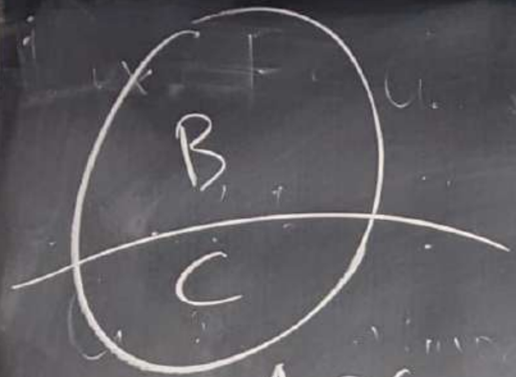
$$G = \bigsqcup_i A_i, \text{ with } \lambda(A_i) < \infty.$$

In each A_i, put $\mathrm{Pois}(t\lambda(A_i))$ many points independently and uniformly at random



$$\prod_i \mathbb{N}_0 \times A_i^{\mathbb{N}} \longrightarrow \mathbb{M}$$

__Fact__ Does not depend on choice of $\mathcal{P}$.



$A \subseteq G$

Let Π be Poisson$(t\lambda(A))$ many pts indep and uniformly at random

What is $N_B(\Pi)$?

$$\mathbb{P}[N_B(\Pi)=k] = \sum_{i \geq k} \mathbb{P}[N_B(\Pi)=k \mid N_A(\Pi)=i] \cdot \mathbb{P}[N_A(\Pi)=i]$$

$$= \sum_{i \geq k} \binom{i}{k} \cdot \left(\frac{\lambda(B)}{\lambda(A)}\right)^k \cdot \left(\frac{\lambda(C)}{\lambda(A)}\right)^{i-k} \cdot e^{-t\lambda(A)} \cdot \frac{(t\lambda(A))^i}{i!}$$

$$= \sum_{i \geq k} \frac{1}{k!\,(i-k)!} \lambda(B)^k \lambda(C)^{i-k} \cdot e^{-t\lambda(B)} e^{-t\lambda(C)} \cdot t^k \cdot t^{i-k} =$$

$[N_A(\Pi) = i]$

$i-k$

$=$

$$e^{-t\lambda(B)} \cdot \frac{(t\lambda(B))^k}{k!} \cdot \cancel{e^{-t\lambda(C)}} \cdot \cancel{\sum_{j=0}^{\infty} \frac{1}{j!}(t\lambda(C))^j} \cdot \boxed{X}$$

What is $\left(N_B(\Pi), N_C(\Pi)\right)$? They are independent.

i.e. $\mathbb{P}[N_B(\Pi) = i, N_C(\Pi) = j] = \boxed{\mathbb{P}[N_B(\Pi) = i]} \cdot \mathbb{P}[N_C(\Pi) = j]$

Exercise.

Fact: $G \curvearrowright (\mathbb{M}, \mathbb{P}_t)$ is ergodic $\iff$ G noncompact

- The Poisson action(s) are essentially free.

Proof Want $\mathbb{P}[\mathrm{Stab}(\Pi) \neq 1] = 0$ (assume not). Choose $R>0$ s.t. $\mathbb{P}[\mathrm{Stab}(\Pi) \cap B(0,R) \neq 1 \text{ and } |\Pi \cap B(0,R)| \geq 2] > 0$

$B(0,2R)$

with pos. prob. $\exists\, x,y \in \Pi \cap B(0,R)$ and $g \in \mathrm{Stab}(\Pi) \cap B(0,R)$

$z_1 = x \qquad z_3 = gx \qquad z_4 = g z_2$

$z_2 = y \qquad z_4 = gy \qquad g = z_4 z_2^{-1}$

$$\boxed{z_3 = z_4 z_2^{-1} z_1}$$

Hence $0 < \mathbb{P}[\exists \underbrace{z_1, z_2, z_3, z_4 \in B(0, 2R) \cap \Pi \text{ and}}_{E} z_3 = z_4 z_2^{-1} z_1]$ -

a contradiction

$$\mathbb{P}[E] = \sum_{k \geq 4} \mathbb{P}[E \mid |B(0, 2R) \cap \Pi| = k] \cdot \mathbb{P}[\ldots]$$